

Phase correlations and quasicondensate in a two-dimensional ultracold Fermi gas

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Abstract

The interplay between dimensionality, coherence and interaction in superfluid Fermi gases is analyzed by the phase correlation function of the field of fermionic pairs. We calculate this phase correlation function for a two-dimensional superfluid Fermi gas with s -wave interactions within the Gaussian pair fluctuation formalism. The spatial behavior of the correlation function is shown to exhibit a rapid (exponential) decay at short distances and a characteristic algebraic decay at large distances, with an exponent matching that expected from Berezinskii-Kosterlitz-Thouless theory of 2D Bose superfluids. We conclude that the Gaussian pair fluctuation approximation is able to capture the physics of quasi long-range order in two-dimensional Fermi gases.

Keywords: Low-dimensional quantum gases, quasicondensate, Correlations

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1. Introduction

The low-temperature physics of interacting quantum gases in reduced dimensions is a subject of intense research. In particular for Bose gases, the physical picture of superfluidity and of the superfluid-to-normal transition has been intensely studied. The Berezinskii-Kosterlitz-Thouless phase transition between the normal and superfluid phases in a trapped atomic Bose gas in two dimensions (2D) has been observed by Hadzibabic *et al.* [1]. A recent series of experimental works focused on the different phases (superfluid, pseudogap, normal) of 2D Bose gases [2, 3, 4, 5]. Fermi gases in 2D have been cooled down into the pseudogap regime [6], but to date the superfluid-to-normal transition has not been observed in the 2D case. However, in 2D Fermi superfluids the interplay between dimensionality, coherence and interaction turns out to be especially interesting since the coherence originates from the interactions, which is not the case in Bose gases. In this contribution, we investigate this interplay through the calculation of the phase correlation functions for the pair field, where the fermionic pairs can be strongly bound Bose-condensed molecules (BEC regime) or weakly bound Cooper pairs (BCS regime).

The physics of superfluidity in uniform two-dimensional Fermi gases at nonzero temperatures is not governed by a true pair condensate, because a true condensed state is destroyed by fluctuations. Rather, the signature of superfluidity is the appearance of long-range phase correlations. The concept of this superfluid state, called a quasicondensate, was developed for Bose gases by Kagan and Popov [7, 8]. In a quasicondensate state, the one-body phase correlation function $F(r)$ decays algebraically at large distances [9],

$$F(r) \propto r^{-\eta}, \quad (1)$$

where $\eta = 1/(n_s \lambda_T^2)$, n_s is the superfluid density, and $\lambda_T = (2\pi/T)^{1/2}$ is the thermal wavelength. The Berezinskii-Kosterlitz-Thouless (BKT) phase transition [10, 11] occurs at a critical temperature $T = T_{BKT}$ that corresponds to a universal value $\eta = 1/4$ [12], when the superfluid density jumps from a finite value to zero. Above T_{BKT} , the correlation function $F(r)$ decays exponentially.

The BKT transition for the ultracold Fermi gases in the BCS-BEC crossover has been theoretically studied using the long-wavelength approximation for the effective action of the bosonic pair field [13, 14, 15]. The phase correlation functions for these bosonic pairs has been considered within the same approximation in Ref. [16] revealing the algebraic decay in agreement with BKT theory. However, within the long-wavelength approximation this algebraic decay of the correlation function $F(r)$ occurs at all r , whereas it is physically expected only at sufficiently large distances. Therefore a treatment beyond the long-wavelength approach is necessary in order to describe phase correlations at large and intermediate distances r , and to estimate a spatial range at which the quasi long-range order appears. For ultracold Bose gases, correlations were theoretically studied using different approaches, e. g., a modified many-body T -matrix theory [17, 18] or Monte Carlo calculations [19]. For the ultracold Fermi gases, to the best of our knowledge, such a treatment beyond the long-wavelength approach has not yet been performed.

In the present work, the phase correlation functions for the pair field are obtained using the Gaussian pair fluctuation effective action within the path integral formalism. This investigation has been inspired by the following reasons. In the long-wavelength treatment of BKT physics in ultracold Fermi gases in two dimensions [13, 14, 15], the long-wavelength hydrodynamic approximation is claimed to be “non-perturbative” contrary to the Gaussian pair fluctuation (GPF) approach (Ref. [20] and its further developments). In other words, it was claimed that the Gaussian pair fluctuation approach cannot describe the quasicondensate. Here we show that this claim is not true. In recent work [21], a systematic long-wavelength expansion of the effective bosonic action has been developed. This expansion is non-perturbative with respect to the pair field. On the one hand, when we assume the fluctuations to be slowly varying and perform the long-wavelength approximation for the GPF fluctuation action, we arrive at a fluctuation action in the hydrodynamic limit. On the other hand, when we represent the pair field as a sum of the uniform saddle-point value and a small fluctuation, and substitute this trial pair field to the long-wavelength action functional of Ref. [21], we must arrive at the same hydrodynamic limit for the fluctuation action. However, the coefficients of that action functional do not change when we assume the fluctuations to be small. We can physically expect that the aforesaid two limiting transitions (a slow varying pair field and small fluctuations) commute. In this case, (1) the phase correlation functions calculated using the GPF action will exhibit an algebraic decay $\propto r^{-\eta}$ at large distances, (2) the values of η will coincide with those following from the long-wavelength theory. A verification of this hypothesis would help us to bridge the gap between the GPF and non-perturbative long-wavelength approaches for the ultracold fermions.

2. Phase correlation function within the Gaussian pair fluctuation approach

The thermodynamics of the ultracold atomic Fermi gases within the Gaussian pair fluctuation formalism is completely determined by the partition function represented through the path integral,

$$\mathcal{Z} \propto \int \mathcal{D}[\varphi^\dagger, \varphi] e^{-S_{fluct}[\varphi^\dagger, \varphi]} \quad (2)$$

where $S_{fluct}[\varphi^\dagger, \varphi]$ is the quadratic fluctuation action functional. The fluctuation action was derived for the ultracold fermions in 3D at $T = T_c$ in Ref. [20], below T_c in Ref. [22], and for imbalanced fermions in Refs. [23, 24]. The fluctuation action for an imbalanced Fermi gas with s -wave pairing in two dimensions has been derived in Ref. [25]. We use this fluctuation action in the present work:

$$S_{fluct} = \sum_{\mathbf{q}, m} \left[M_{1,1}(\mathbf{q}, i\Omega_m) \varphi_{\mathbf{q}, m}^\dagger \varphi_{\mathbf{q}, m} + \frac{1}{2} M_{1,2}(\mathbf{q}, i\Omega_m) (\varphi_{\mathbf{q}, m}^\dagger \varphi_{-\mathbf{q}, -m}^\dagger + \varphi_{\mathbf{q}, m} \varphi_{-\mathbf{q}, -m}) \right], \quad (3)$$

where $M_{j,k}(\mathbf{q}, i\Omega_m)$ are the matrix elements of the inverse pair fluctuation propagator \mathbb{M} ,

$$\begin{aligned} M_{1,1}(\mathbf{q}, i\Omega_m) = & \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{4E_{\mathbf{k}}E_{\mathbf{k}+\mathbf{q}}} \frac{\sinh(\beta E_{\mathbf{k}})}{\cosh(\beta E_{\mathbf{k}}) + \cosh(\beta \zeta)} \\ & \times \left(\frac{(\xi_{\mathbf{k}+\mathbf{q}} + E_{\mathbf{k}+\mathbf{q}})(\xi_{\mathbf{k}} + E_{\mathbf{k}})}{i\Omega_m - E_{\mathbf{k}} - E_{\mathbf{k}+\mathbf{q}}} + \frac{(E_{\mathbf{k}+\mathbf{q}} - \xi_{\mathbf{k}+\mathbf{q}})(\xi_{\mathbf{k}} + E_{\mathbf{k}})}{i\Omega_m - E_{\mathbf{k}} + E_{\mathbf{k}+\mathbf{q}}} \right. \\ & + \frac{(\xi_{\mathbf{k}+\mathbf{q}} + E_{\mathbf{k}+\mathbf{q}})(\xi_{\mathbf{k}} - E_{\mathbf{k}})}{i\Omega_m + E_{\mathbf{k}} - E_{\mathbf{k}+\mathbf{q}}} + \left. \frac{(E_{\mathbf{k}+\mathbf{q}} - \xi_{\mathbf{k}+\mathbf{q}})(\xi_{\mathbf{k}} - E_{\mathbf{k}})}{i\Omega_m + E_{\mathbf{k}} + E_{\mathbf{k}+\mathbf{q}}} \right) \\ & - \frac{1}{g}, \end{aligned} \quad (4)$$

and

$$\begin{aligned} M_{1,2}(\mathbf{q}, i\Omega_m) = & -\Delta^2 \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{4E_{\mathbf{k}}E_{\mathbf{k}+\mathbf{q}}} \frac{\sinh(\beta E_{\mathbf{k}})}{\cosh(\beta E_{\mathbf{k}}) + \cosh(\beta \zeta)} \\ & \times \left(\frac{1}{i\Omega_m - E_{\mathbf{k}} - E_{\mathbf{k}+\mathbf{q}}} - \frac{1}{i\Omega_m - E_{\mathbf{k}} + E_{\mathbf{k}+\mathbf{q}}} \right. \\ & + \left. \frac{1}{i\Omega_m + E_{\mathbf{k}} - E_{\mathbf{k}+\mathbf{q}}} - \frac{1}{i\Omega_m + E_{\mathbf{k}} + E_{\mathbf{k}+\mathbf{q}}} \right), \end{aligned} \quad (5)$$

where $\xi_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m} - \mu$ are the fermion energies counted from the average chemical potential of “spin-up” and “spin-down” fermions $\mu = (\mu_\uparrow + \mu_\downarrow)/2$. Furthermore, $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$ are the Bogoliubov excitation energies, $\zeta = (\mu_\uparrow - \mu_\downarrow)/2$ is a measure of imbalance through the chemical potentials, and $\Omega_m = 2\pi m/\beta$ are the bosonic Matsubara frequencies with $\beta = 1/(k_B T)$. In the chosen system of units, $\hbar = 1$, the fermion mass $m = 1/2$ and the Fermi energy of a free-fermion gas in 2D is $E_F \equiv \pi \hbar^2 n/m = 1$, where n is the total fermion density. The coupling strength g is renormalized through the binding energy of a two-particle bound state E_b , in the same way as in Ref. [26]:

$$\frac{1}{g} = \frac{1}{8\pi} \left(\ln \frac{E_b}{E} + i\pi \right) - \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{2k^2 - E + i\delta}, \quad (6)$$

with δ a positive infinitesimal.

Within the Gaussian pair fluctuation approach [20, 22, 23, 24, 25] the action is expanded quadratically around a (uniform) saddle-point Δ . Therefore the amplitude and the phase components of the fluctuation field variables can be written as follows:

$$\begin{aligned}\varphi(\mathbf{r}, \tau) &= a(\mathbf{r}, \tau) + i\Delta \cdot \theta(\mathbf{r}, \tau), \\ \varphi^\dagger(\mathbf{r}, \tau) &= a(\mathbf{r}, \tau) - i\Delta \cdot \theta(\mathbf{r}, \tau).\end{aligned}$$

The fluctuation coordinates in the coordinate/time representation $\varphi(\mathbf{r}, \tau)$ are expressed through their Fourier amplitudes entering (3):

$$\varphi(\mathbf{r}, \tau) = \frac{1}{L\sqrt{\beta}} \sum_{\mathbf{q}} \sum_{m=-\infty}^{\infty} e^{i\mathbf{q}\cdot\mathbf{r} - i\Omega_m\tau} \varphi_{\mathbf{q},m}, \quad (7)$$

$$\varphi^\dagger(\mathbf{r}, \tau) = \frac{1}{L\sqrt{\beta}} \sum_{\mathbf{q}} \sum_{m=-\infty}^{\infty} e^{-i\mathbf{q}\cdot\mathbf{r} + i\Omega_m\tau} \varphi_{\mathbf{q},m}^\dagger. \quad (8)$$

The Fourier components of the amplitude and phase are expressed through $\varphi_{\mathbf{q},m}$ as:

$$a_{\mathbf{q},m} = \frac{\varphi_{\mathbf{q},m} + \varphi_{-\mathbf{q},-m}^\dagger}{2}, \quad \theta_{\mathbf{q},m} = \frac{\varphi_{\mathbf{q},m} - \varphi_{-\mathbf{q},-m}^\dagger}{2i\Delta}. \quad (9)$$

Note that although $\theta(\mathbf{r}, \tau)$ is an angular field, we do not take this periodicity into account, as usual in this formalism.

Let us introduce the matrix elements which are even (e) and odd (o) with respect to Ω_m :

$$\begin{aligned}M_{1,1}^{(e)}(q, i\Omega_m) &= \frac{1}{2} [M_{1,1}(q, i\Omega_m) + M_{1,1}(q, -i\Omega_m)], \\ M_{1,1}^{(o)}(q, i\Omega_m) &= \frac{1}{2} [M_{1,1}(q, i\Omega_m) - M_{1,1}(q, -i\Omega_m)].\end{aligned} \quad (10)$$

The matrix element $M_{1,2}(q, i\Omega_m)$ is even. The action functional for the Gaussian fluctuations is then rewritten in terms of the amplitude and phase fluctuations:

$$\begin{aligned}S_{fluct} &= \sum_{\mathbf{q},m} \left\{ [M_{1,1}^{(e)}(q, i\Omega_m) + M_{1,2}(q, i\Omega_m)] a_{\mathbf{q},m}^\dagger a_{\mathbf{q},m} \right. \\ &\quad + [M_{1,1}^{(e)}(q, i\Omega_m) - M_{1,2}(q, i\Omega_m)] \Delta^2 \theta_{\mathbf{q},m}^\dagger \theta_{\mathbf{q},m} \\ &\quad \left. + M_{1,1}^{(o)}(q, i\Omega_m) i\Delta (a_{\mathbf{q},m}^\dagger \theta_{\mathbf{q},m} - \theta_{\mathbf{q},m}^\dagger a_{\mathbf{q},m}) \right\}. \quad (11)\end{aligned}$$

The analogous amplitude-phase representation was considered for the fluctuation action of a balanced Fermi gas in 3D below T_c , Ref. [22]. In general, the amplitude and phase fluctuations are coupled in (11). They are decoupled only for $\Omega_m = 0$, because $M_{1,1}^{(o)}(q, 0) = 0$.

The fluctuation action is quadratic, so that the correlation functions of the field variables are calculated in a straightforward way. We need to determine the amplitude-amplitude, phase-phase and amplitude-phase quadratic correlation functions. They are expressed through the matrix

elements as follows:

$$\langle a_{\mathbf{q},m} a_{\mathbf{q}',m'}^\dagger \rangle = \delta_{\mathbf{q}',\mathbf{q}} \delta_{m',m} \frac{1}{2} \frac{M_{1,1}^{(e)}(q, i\Omega_m) - M_{1,2}(q, i\Omega_m)}{\det \mathbb{M}(q, i\Omega_m)}, \quad (12)$$

$$\langle \theta_{\mathbf{q},m} \theta_{\mathbf{q}',m'}^\dagger \rangle = \delta_{\mathbf{q}',\mathbf{q}} \delta_{m',m} \frac{1}{2\Delta^2} \frac{M_{1,1}^{(e)}(q, i\Omega_m) + M_{1,2}(q, i\Omega_m)}{\det \mathbb{M}(q, i\Omega_m)}, \quad (13)$$

$$\langle a_{\mathbf{q},m} \theta_{\mathbf{q}',m'}^\dagger \rangle = -\delta_{\mathbf{q}',\mathbf{q}} \delta_{m',m} \frac{i}{2\Delta} \frac{M_{1,1}^{(o)}(q, i\Omega_m)}{\det \mathbb{M}(q, i\Omega_m)}, \quad (14)$$

$$\langle a_{\mathbf{q},m}^\dagger \theta_{\mathbf{q}',m'} \rangle = \delta_{\mathbf{q}',\mathbf{q}} \delta_{m',m} \frac{i}{2\Delta} \frac{M_{1,1}^{(o)}(q, i\Omega_m)}{\det \mathbb{M}(q, i\Omega_m)}. \quad (15)$$

We consider the phase fluctuation correlation functions

$$F(\mathbf{r} - \mathbf{r}', \tau - \tau') \equiv \langle e^{i\theta(\mathbf{r},\tau)} e^{-i\theta(\mathbf{r}',\tau')} \rangle_{S_{fluct}}. \quad (16)$$

For a Gaussian fluctuation action, Wick's decomposition theorem allows to express the correlation function (16) as

$$F(\mathbf{r}, \tau) = e^{-G(\mathbf{r},\tau)} \quad (17)$$

with the quadratic phase correlator,

$$G(\mathbf{r}, \tau) \equiv \frac{1}{2} \langle [\theta(\mathbf{r}, \tau) - \theta(0, 0)]^2 \rangle_{S_{fluct}}. \quad (18)$$

The analogous treatment for Bose gases in 2D was performed in Refs. [17, 18], and for a Fermi gas in 2D within the long-wavelength approximation in Ref. [16].

Using the Fourier expansion for the phase and the obtained quadratic correlators, we arrive at the result:

$$G(r, \tau) = \frac{1}{4\pi\Delta^2} \int_0^\infty q dq \frac{1}{\beta} \sum_{m=-\infty}^\infty [1 - J_0(qr) e^{-i\Omega_m \tau}] \times \frac{M_{1,1}^{(e)}(q, i\Omega_m) + M_{1,2}(q, i\Omega_m)}{\det \mathbb{M}(q, i\Omega_m)}. \quad (19)$$

In order to analyze the instantaneous behavior of the phase, we consider the case $\tau = 0$ and the functions $F(r) \equiv F(r, 0)$ and $G(r) \equiv G(r, 0)$. The bosonic Matsubara summations are performed using contour integrations analogously to Refs. [20, 22, 23, 24, 25]. The result is:

$$G(r) = \frac{1}{4\pi\Delta^2} \int_0^\infty q dq [1 - J_0(qr)] S(q), \quad (20)$$

with $\delta \rightarrow +0$ and with the spectral function

$$S(q) = \frac{1}{\pi} \int_{-\infty}^\infty d\omega \frac{1}{1 - e^{-\beta\omega}} \text{Im} \left(\frac{M_{1,1}(q, \omega + i\delta) + M_{1,2}(q, \omega + i\delta)}{\det \mathbb{M}(q, \omega + i\delta)} \right). \quad (21)$$

The integral over q in (20) is free from the long-wavelength divergence at $q \rightarrow 0$ due to the factor $1 - J_0(qr)$, but an ultraviolet divergence at $q \rightarrow \infty$ appears, similarly as in Refs. [17, 18].

Namely, the spectral function $S(q)$ decays as q^{-2} at $q \rightarrow \infty$. Therefore the integral $\int_0^\infty S(q) q dq$ in (20) diverges logarithmically at the upper bound, while the other integral, $\int_0^\infty J_0(qr) S(q) q dq$, converges.

The divergent integral $\int_0^\infty S(q) q dq$ does not depend on the distance r . Hence this divergence leads to an infinite factor which does not depend on r and hence does not influence the decay rate of the correlation function. Because this factor is one and the same for all r , the spatial behavior of the correlation functions can be analyzed considering the relative (fractional) correlation function

$$F^R(r, r_c) \equiv \frac{F(r)}{F(r_c)} \quad (22)$$

with an arbitrary r_c . This fractional correlation function is expressed as

$$F^R(r, r_c) = e^{-G^R(r, r_c)} \quad (23)$$

with the quadratic phase correlator,

$$G^R(r, r_c) = \frac{1}{4\pi\Delta^2} \int_0^\infty q dq [J_0(qr_c) - J_0(qr)] S(q). \quad (24)$$

As follows immediately from (22), the fractional correlation function $F^R(r, r_c)$ at $r = 0$ becomes equal to $1/F(r_c)$, so that the aforesaid ultraviolet divergence again appears at $r = 0$. The divergence of $F^R(r, r_c)$ at $r = 0$ is then an artifact of the regularization used here. This is a common feature with the correlation function derived in Ref. [16] where $F(r) \propto r^{-\eta}$ for all r . In the present treatment the correlation function increases at $r \rightarrow 0$ logarithmically, i. e., more slowly than within the long-wavelength approximation.

The physical reason of the ultraviolet divergence discussed above is the restriction of the treatment to the Gaussian fluctuations about the saddle point. A convergent integral over the momentum q might be obtained by a (partial) series summations over higher-order terms in powers of the fluctuations. A complete regularization of the correlation function including the point $r = 0$ is beyond the scope of the present work.

It can be shown that the long-range behavior of the correlation functions can be insensitive to the concrete way of regularization of the integral in $G(r)$. Let us assume that the spectral function $S(q)$ is renormalized, $S(q) \rightarrow S^{reg}(q)$, in such a way that the integral

$$G^{reg}(r) = \frac{1}{4\pi\Delta^2} \int_0^\infty q dq [1 - J_0(qr)] S^{reg}(q) \quad (25)$$

converges. We assume also that the renormalized spectral function $S^{reg}(q)$ decays faster than $S(q)$ at large q but tends to $S(q)$ at small q . At large distances, $J_0(qr)$ is small, and hence we arrive at the logarithmic small q divergence in (25) at $r \rightarrow \infty$. Therefore the small q range is crucial for the increase of $G^{reg}(r)$ at large r . As long as $S^{reg}(q) \rightarrow S(q)$ at small q , $G^{reg}(r)$ is not sensitive to a behavior of $S^{reg}(q)$ at large q .

In order to verify this reasoning, we consider a simple alternative method of regularization for the correlation function $G(r)$ introducing an ultraviolet cutoff q_c for the momentum q , so that the regularized correlation function is

$$G^{reg}(r, q_c) = \frac{1}{4\pi\Delta^2} \int_0^{q_c} q dq [1 - J_0(qr)] S(q). \quad (26)$$

The regularized correlation function

$$F^{reg}(r, q_c) = e^{-G^{reg}(r, q_c)} \quad (27)$$

is convergent for all r .

As shown above, the asymptotic behavior of the correlation functions at sufficiently large r is determined by the spectral function in the small q region. Therefore the cutoff regularization should not influence this asymptotic behavior. We thus expect that the fractional correlation functions (22) and the regularized correlation functions with (26) decay in the same way at large r . This conclusion will be numerically verified in the next section.

3. Results and discussion

In this section the spatial profile of the phase correlation function is discussed for different temperatures and binding energies. It is especially interesting to compare the decay of the phase correlations described by formulae (23) and (24) with the algebraic decay for a quasicondensate with the power index $\eta = 1/(n_s \lambda_T^2)$ following from the BKT theory.

Within the microscopic BKT theory of the superfluidity for ultracold fermions in two dimensions, the parameters of the superfluid density $n_s(T, \mu, \zeta, \Delta)$ (the chemical potentials μ, ζ and the gap Δ) are determined from a joint solution of the gap equation and equations normalizing the total fermion density n and the density difference $\delta n = n_\uparrow - n_\downarrow$ (the number equations). It should be noted that the superfluid density entering the long-wavelength action functionals [13, 14, 15] as a prefactor at $(\nabla\theta)^2$ (e. g., formula (22) of Ref. [15]),

$$n_s(T, \mu, \zeta, \Delta) = \frac{1}{4\pi} \int_0^\infty dk k \left\{ 1 - \frac{\xi_k}{E_k} \frac{\sinh(\beta E_k)}{\cosh(\beta E_k) + \cosh(\beta \zeta)} - k^2 \beta \frac{\cosh \beta E_k \cosh(\beta \zeta) + 1}{[\cosh \beta E_k + \cosh(\beta \zeta)]^2} \right\}. \quad (28)$$

is the *mean-field* expression, because the fluctuation correction to the mean-field action up to quadratic order is already contained in $(\nabla\theta)^2$.

If one accounts for fluctuations through the chemical potentials entering the superfluid density, n_s must be necessarily completed with a fluctuation contribution as $n_s^{(tot)} = n_s^{(mean-field)} + n_s^{(fluct)}$. However, the fluctuation contribution $n_s^{(fluct)}$ is not present as a prefactor of a fluctuation field in the quadratic Gaussian action: it can only appear as a fluctuation field prefactor in the next (quartic) order correction to S_{fluct} . Thus an account of the influence of fluctuations on the prefactors in the Gaussian fluctuation action would be, strictly speaking, beyond the quadratic approximation. Hence it is consistent to calculate the mean-field superfluid density (28) with the parameters determined using the mean-field number equations. This principle is held in all known works on the microscopic BKT theory, including Refs. [13, 14, 15]. For the same reason, in order to adequately compare the decay of the correlation functions derived in Section 2 with that following from the microscopic BKT theory, the matrix elements $M_{j,k}$ must be calculated with the mean-field values of the chemical potentials.

In Fig. 1, the phase correlation functions are plotted for the Fermi gas at different temperatures in 2D, for binding energies $E_b = 0.1 E_F$ (BCS regime) and $E_b = E_F$ (BEC-BCS crossover regime). The behavior of the phase correlation function qualitatively agrees with the result of

Ref. [19] for a Bose gas in 2D using the Monte Carlo technique. At small and intermediate distances, the correlation function decreases rapidly, obeying closely an exponential decay law. At sufficiently large distances, the correlation function decreases much slower, and the decay tends to a power law.

In order to characterize the power-law decay, we analyze the parameter α determined through the logarithmic derivative of the phase correlation function, defined as

$$\alpha(r) \equiv -\frac{r}{F(r)} \frac{\partial F(r)}{\partial r} = r \frac{\partial G(r)}{\partial r}. \quad (29)$$

Note that when $F(r) \propto r^{-\eta}$, we obtain $\alpha(r) = \eta$.

The derivative is determined straightforwardly using formula (24):

$$\alpha(r) = \frac{r}{4\pi\Delta^2} \int_0^\infty q^2 dq J_1(qr) \times \frac{1}{\pi} \int_{-\infty}^\infty d\omega \operatorname{Im} \left(\frac{1}{1 - e^{-\beta(\omega+i\delta)}} \frac{M_{1,1}(q, \omega + i\delta) + M_{1,2}(q, \omega + i\delta)}{\det \mathbb{M}(q, \omega + i\delta)} \right). \quad (30)$$

As distinct from the phase correlation function, the integral in (30) is convergent and does not require regularization.

In Fig. 2, the parameter $\alpha(r)$ determined by formula (30) for a Fermi gas in 2D is plotted at different temperatures with the same values of the binding energy as in Fig. 1. For comparison, also the parameter $\eta(T)$ from the microscopic BKT theory is shown (dot-dashed lines).

The magnitude of $\alpha(r)$ can be a measure of the decay rate for the phase correlations. We can see that the phase correlation function rapidly falls down at intermediate distances. For sufficiently large r , the parameter $\alpha(r)$ explicitly turns to η , as expected. This trend is an indication of long-range correlations in a two-dimensional Fermi gas, and confirms the suggestion that the Gaussian pair formalism is capable to adequately describe the quasicondensate phase.

The spatial dependence of the decay rate depends relatively weakly on the binding energy, but is temperature dependent. At small and intermediate distances, where the decay is exponential, the correlation functions depends only very weakly on the temperature. At distances where the decay of phase correlations is algebraic, there is a strong temperature dependence. The crossover value r_q where the fast decrease of the correlation function changes into an algebraic decay can be interpreted as the characteristic distance at which quasi long-range order is formed. When the temperature is close to T_{BKT} , we can estimate $r_q \gtrsim 1/k_F$. With decreasing temperature the distance r_q gradually rises.

In order to numerically verify the above conclusion that the long-range decay of the phase correlation functions is not sensitive to a choice of the regularization, we perform the numeric study of the regularized correlation function $F^{reg}(r, q_c)$. Fig. 3 shows the phase correlation functions $F^{reg}(r, q_c)$ obtained using the cutoff regularization of the ultraviolet divergence with $q_c = 20k_F$ for the binding energy $E_b = 0.1E_F$ in the linear (a) and logarithmic (b) scales. As distinct from the fractional correlation functions $F^R(r, r_c)$, the cutoff-regularized correlation functions contain no divergence anywhere and turn to unity at $r = 0$. The coordinate dependence of $F^{reg}(r, q_c)$ at intermediate and large distances is very similar to that of $F^R(r, r_c)$: the function $F^{reg}(r, q_c)$ decays almost exponentially at intermediate distances, as seen from Fig. 3 (a), and exhibits an algebraic decay at large distances, as follows from Fig. 3 (b). To check this similarity quantitatively, we plot the ratio $F^{reg}(r, q_c)/F^R(r, r_c)$ in Fig. 4. For small r , the regularized correlation function $F^{reg}(r, q_c)$ oscillates due to the factor $J_0(q_c r)$ at the upper bound of the

integral over q . With increasing r , these oscillations gradually fall down, vanishing at long distances, and the ratio $F^{reg}(r, q_c)/F^R(r, r_c)$ tends to a constant value. The numerical check therefore confirms the conclusion obtained above analytically: despite different regularizations, the long-range decay is one and the same for $F^{reg}(r, q_c)$ and $F^R(r, r_c)$.

4. Conclusions

We have derived the phase correlation function for an ultracold Fermi gas in two dimensions on the basis of the Gaussian pair fluctuation action without assuming the fluctuation field slowly varying. The resulting correlation function describes the decay of phase correlations in the whole range of r , revealing a fast decrease at intermediate distances and the characteristic algebraic decay at large distances. This algebraic decay obtained for the correlation functions of the phase of the fermion pair field excellently matches the power law following from the microscopic BKT theory for Bose gases. The appearance of the algebraic long-range order shows that the existence of quasicondensate in two-dimensional Fermi gases can be adequately described within the Gaussian pair fluctuation approach.

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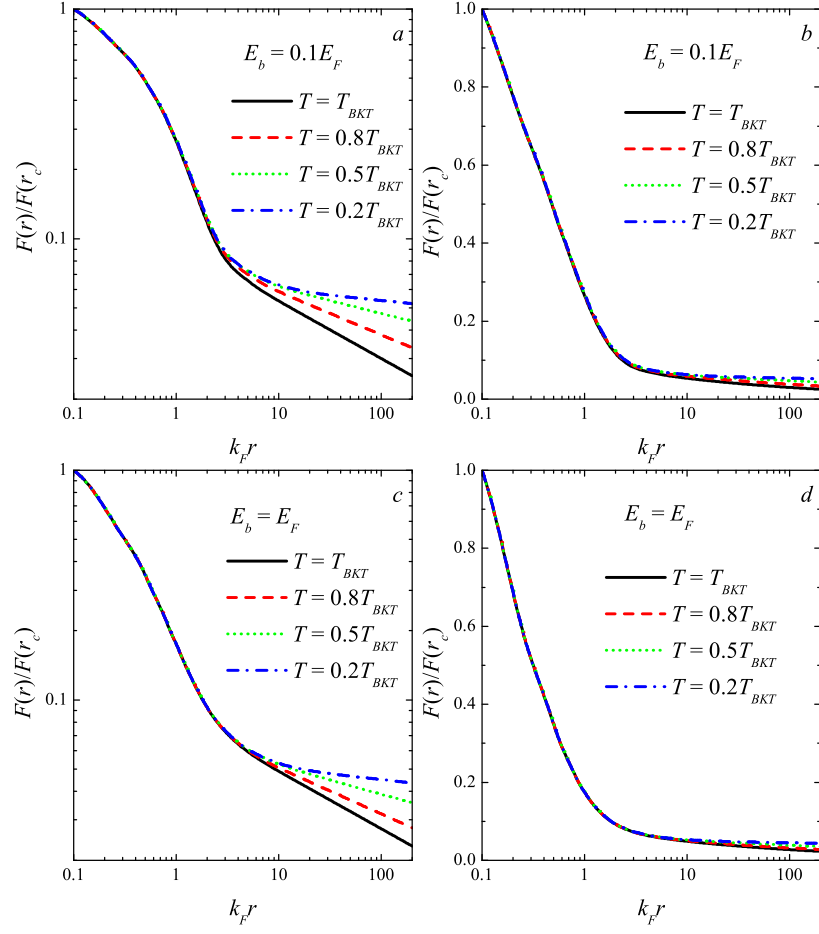


Figure 1: Phase correlation functions for the Fermi gas in 2D with the binding energy $E_b = 0.1 E_F$ (a, b) and $E_b = E_F$ (c, d). In panels a and c, the results are represented on a logarithmic scale for both x and y axes. In panels b and d, the logarithmic scale is used for the x axis only.

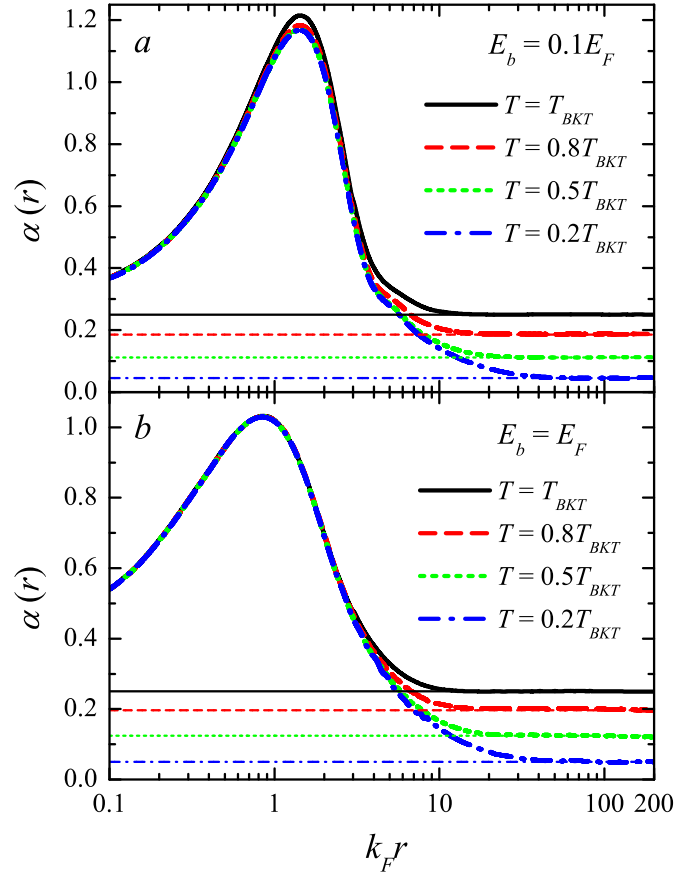


Figure 2: The function $\alpha(r) = r \partial G(r) / \partial r$ for a Fermi gas in 2D with the binding energy $E_b = 0.1 E_F$ (a) and $E_b = E_F$ (b) at different temperatures. The parameter η calculated according to the BKT theory is shown by dot-dashed lines.

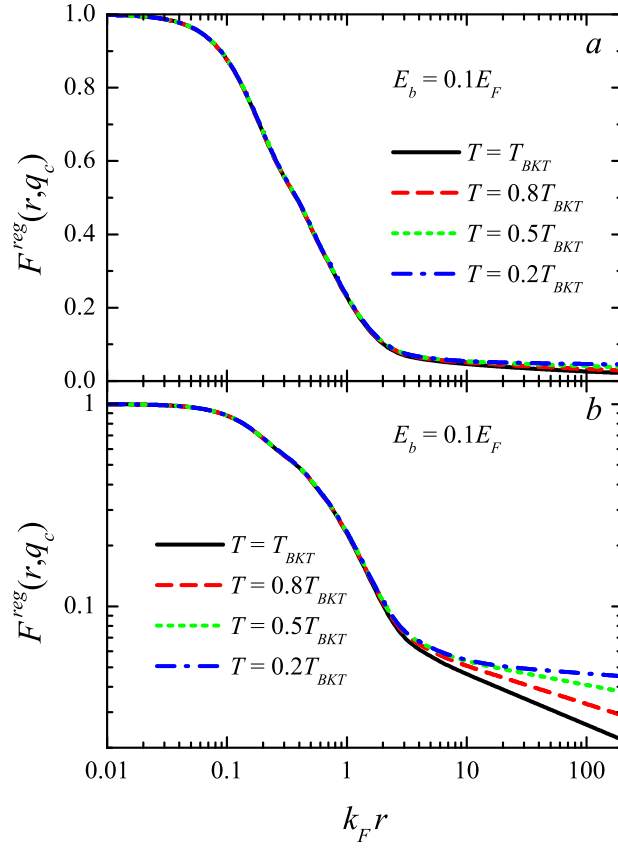


Figure 3: Cutoff-regularized phase correlation functions for the Fermi gas in 2D with the binding energy $E_b = 0.1E_F$ on the linear scale (a) and on the logarithmic scale (b).

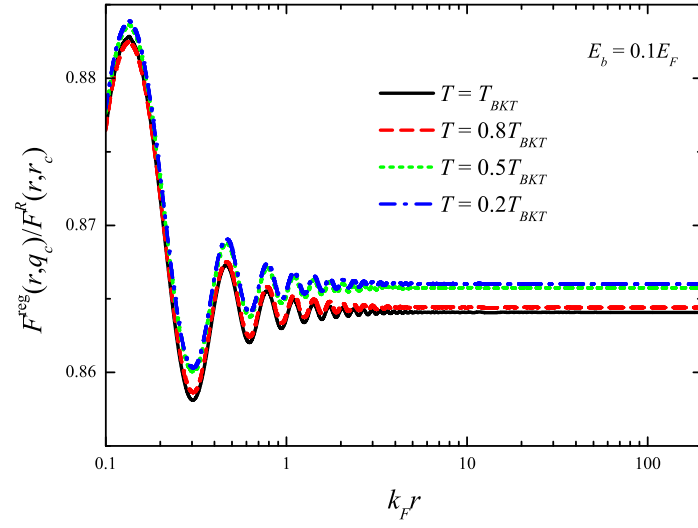


Figure 4: Ratio of the phase correlation functions $F^{reg}(r, q_c)/F^R(r, r_c)$ for the binding energy $E_b = 0.1E_F$ and different temperatures.